Question

A one-meter stick is cut into c random places. What is the expected length of the smallest of the c+1 pieces created?

Answer

$$\frac{1}{(c+1)^2}$$

Solution

Let's define n as the number of pieces formed. With c cuts, the n=c+1.

First, let's solve for the probability that the smallest piece will be less than a specified value. For example, with n=3, what is the probability the smallest piece will be under 0.05.

A way we could achieve this would be to set aside 3*0.05=0.15 of the board to increase the size of each of the three pieces by 0.05. As long as the two cuts were in the other 85% of the board we could pad the three pieces by 0.05 each to ensure none were less than 0.05. The chances that two cuts are both in that 85% section is 0.85^2 = 0.7225. That alternative is at least one cut was in the 15% section meant for padding, resulting in a piece under 0.05. That probability is 1-0.85^2 = 0.2775.

The expected value of a random variable is the integral from 0 to the maximum value of that variable of the probability that it hasn't happened yet.

In the case of two cuts, the maximum length of the shortest piece is 1/3. Recall that n is the number of pieces formed. An expression for the expected value of the shortest piece is:

$$\int_0^{1/3} (1-3x)^2 \, dx$$

Integrating this results in a value of $(\frac{1}{3})^2 = \frac{1}{9}$

The general formula for n pieces is:

$$\int_0^{1/n} (1-nx)^{n-1} \, dx$$

You can do the calculus yourself or use an integral calculator (I like the one at www.integral-calculator.com) to see that this evaluates to:

$$\frac{1}{n^2}$$

I would like to thank WizardOfVegas.com forum member Ace2 for this solution. He puts it in his own words here: https://wizardofvegas.com/forum/questions-and-answers/math/34502-easy-math-puzzles/104/#post964835